

APPROXIMATIONS OF GROUPS, CHARACTERIZATIONS OF SOFIC GROUPS, AND EQUATIONS OVER GROUPS.

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Abstract

We give new characterizations of sofic groups:

- A group G is sofic if and only if it is a subgroup of a quotient of a direct product of alternating or symmetric groups.
- A group G is sofic if and only if any system of equations solvable in all alternating groups is solvable over G .

The last characterization allows to express soficity of an existentially closed group by $\forall\exists$ -sentences.

Keywords: sofic groups, approximations, equations over groups.

1 Introduction

Sofic groups have been defined¹ in [15, 29] in relation with the Gottschalk surjunctivity conjecture. Hyperlinear groups have been introduced in [25] in relation with Connes’ embedding conjecture. It is known that sofic groups are hyperlinear, the reverse inclusion is an open question.

Some famous group theory conjectures (Kervaire-Laudenbach, Gottschalk, Connes’ embedding conjectures) are established for sofic groups, see [10, 24] and the references therein. It is also known that some important classes of groups are sofic, for example, amenable, residually amenable, extensions of sofic groups by amenable groups, etc., [10, 12, 24]. An open question is whether or not all groups are sofic (respectively, hyperlinear).

Classically, sofic (resp. hyperlinear) groups are defined as being metric approximable by symmetric groups (resp. finite-dimensional unitary groups), [28]. It is possible to define metric approximations by different classes of groups, see [2] and Definition 3 of the present paper. We call them $(\mathcal{K}, \mathcal{L})$ -approximations, where \mathcal{K} is a class of groups and \mathcal{L} is a class of invariant length functions on these groups. Choosing the \mathcal{K} and \mathcal{L} one defines different classes of groups as $(\mathcal{K}, \mathcal{L})$ -approximable groups. For example, weakly sofic groups, [13], and linear sofic groups, [3], for $(\mathcal{K}, \mathcal{L}) = (\{\text{all finite groups}\}, \{\text{all length functions}\})$ and $(\mathcal{K}, \mathcal{L}) = (\{GL_n(\mathbb{C})\}, \{\text{normalized rank}\})$, respectively. The following result is classical here, see Proposition 1 for details.

- A group G is $(\mathcal{K}, \mathcal{L})$ -approximable if and only if G is isomorphic to a subgroup of a metric ultraproduct with respect to \mathcal{L} of groups from \mathcal{K} .

By definition, the metric approximation depends on invariant length functions and a class of groups. The structure of the set of invariant length functions on a group depends on the algebra of the conjugacy classes of this group. Indeed, an invariant length function $\|\cdot\|$, see

¹Definition 3.5 of [1] is the definition of sofic groups where the authors missed the separability conditions. All groups trivially satisfy this definition.

Definition 2, is constant along conjugacy classes and satisfies inequality $\|g_3\| \leq \|g_1\| + \|g_2\|$ for $g_3 \in C(g_1)C(g_2)$, where $C(g)$ is the conjugacy class of g .

In the present article we investigate the notion of approximation based on products of conjugacy classes without direct use of length functions, see Definition 6. Such approximations will be called \mathcal{K} -approximations². A group possessing \mathcal{K} -approximation will be called \mathcal{K} -approximable group. Let Sym , Alt , Nil , Sol , Fin be the classes of finite symmetric, finite alternating, finite nilpotent, finite solvable and all finite groups, respectively. We show that the classes of Alt -approximable groups, Sym -approximable groups, and sofic groups coincide. Fin -approximable groups are called weakly sofic [13].

Proposition 1 with Lemma 3 implies that a \mathcal{K} -approximable group is isomorphic to a subgroup of a quotient of an (unrestricted) direct product of groups from \mathcal{K} . Is it true that any quotient of any direct product of groups from \mathcal{K} is \mathcal{K} -approximable? We don't know the answer to the question but we answer it affirmatively if \mathcal{K} is a class of compact groups satisfying $\mathcal{K} = \prod(\mathcal{K})$, where $\prod(\mathcal{K})$ is a class of all finite direct products of groups from \mathcal{K} . The question has an affirmative answer for $\mathcal{K} = Sym$ and Alt also. We start by proving that for a class of compact groups \mathcal{K} a quotient of a direct product $G = \prod_{i \in \mathbb{N}} H_i$, $H_i \in \mathcal{K}$, is $\prod(\mathcal{K})$ -approximable, see Proposition 5. Then we show that $\mathcal{K}.approx = \prod(\mathcal{K}).approx$ for $\mathcal{K} = Alt$ and Sym , Proposition 9. The property $\mathcal{K}.approx = \prod(\mathcal{K}).approx$ allow us to give the following characterization of \mathcal{K} -approximable groups.

. Let $F = \langle a_1, \dots, a_r \rangle$ be a finitely generated free group and $N \triangleleft F$. Then F/N is Nil , Sol , Fin -approximable if and only if $N = \langle\langle N \rangle\rangle_{\hat{F}} \cap F$. Here $F \hookrightarrow \hat{F}$ and \hat{F} is the pro-nilpotent, pro-solvable, pro-finite completion of F , respectively. $\langle\langle N \rangle\rangle_{\hat{F}}$ denotes the minimal normal subgroup of the corresponding \hat{F} containing N . (Notice, that $\langle\langle N \rangle\rangle_{\hat{F}}$ need not to be topologically closed.)

The similar characterization for sofic groups has a peculiarity due to the fact that Alt and Sym are not closed under taking subgroups. Still, one can define $F \hookrightarrow \mathcal{A}$ (resp. $F \hookrightarrow \mathcal{S}$) in such a way that F/N is sofic if and only if $N = \langle\langle N \rangle\rangle_{\mathcal{A}} \cap F$ (resp. $N = \langle\langle N \rangle\rangle_{\mathcal{S}} \cap F$). The group \mathcal{A} , for example, is a direct product of alternating groups, every alternating group A_m appears finitely many times, one for each homomorphism $\phi : F \rightarrow A_m$, with the inclusion $F \hookrightarrow \mathcal{A}$ naturally defined by these ϕ 's. Notice, that in contrast to the above statement, the closure of F (in the product Tychonoff topology) is not the whole \mathcal{A} . In fact, this closure is isomorphic to the pro-finite completion of F . The construction of \mathcal{S} is similar the only difference that instead of alternating groups A_m one should take symmetric groups S_m .

We may reformulate the above characterization using equations over groups. Let $\bar{a} = (a_1, \dots, a_r)$ and $\bar{x} = (x_1, \dots, x_k)$ be the symbols for constants and variables, respectively. $|\bar{a}| = r$ and $|\bar{x}| = k$. Let $\langle\bar{a}, \bar{x}\rangle$ be a free group freely generated by \bar{a} and \bar{x} . Let $w_i \in \langle\bar{a}, \bar{x}\rangle$ and $\bar{w} = (w_1, \dots, w_n)$. Consider the system of equations $\bar{w} = 1$ ($w_1 = 1, \dots, w_n = 1$).

Definition 1. We say that \bar{w} is solvable in a group G if the sentences

$$\forall \bar{a} \exists \bar{x} \bigwedge_{i=1}^n w_i(\bar{a}, \bar{x}) = 1$$

is valid in G . We say that a system \bar{w} is solvable over group G if for some $H > G$

$$\forall \bar{a} \in G^r \exists \bar{x} \in H^k \bigwedge_{i=1}^n w_i(\bar{a}, \bar{x}) = 1$$

²Some other types of approximations are defined in [1, 2].

Denote by $Sys(G)$ the set of all finite systems of equations solvable in G . For a class \mathcal{K} let $Sys(\mathcal{K}) = \bigcap_{G \in \mathcal{K}} Sys(G)$. In Section 4 the following proposition is proved.

Proposition. *Let $\mathcal{K} = Nil, Sol, Fin, Alt$ or Sym . Then a group G is \mathcal{K} -approximable if and only if all systems $\bar{w} \in Sys(\mathcal{K})$ are solvable over G .*

Remark.

- It is not clear if the sets $Sys(\mathcal{K})$ are recursive.
- Let F be a noncommutative free group. By the solution of Tarski problem $Sys(F)$ is independent of F and recursive, [18, 26]. It easily follows that $Sys(F) \subset Sys(G)$ for any group G . So, for our purpose $Sys(F)$ are “trivial equations” and it suffices to consider $Sys'(\mathcal{K}) = Sys(\mathcal{K}) \setminus Sys(F)$.
- $Sys(Fin) \neq \emptyset$, see [11].

Although the characterization looks impractical, we can derive some consequences.

Paul Schupp suggests that existentially closed groups are non sofic [14]. The existentially closed groups are analogues of algebraically closed fields, see [16]. The following proposition supports this hypothesis.

Proposition. *Let $\mathcal{K} = Nil, Sol, Alt, Sym$, or Fin . An existentially closed group is \mathcal{K} -approximable if and only if all groups are \mathcal{K} -approximable.*

Proof. An existentially closed group G is \mathcal{K} -approximable if and only if the sentences

$$\forall \bar{a} \exists \bar{x} \bigwedge_{w \in \bar{w}} w(\bar{a}, \bar{x})$$

are valid in G for any $\bar{w} \in Sys(\mathcal{K})$. A $\forall \exists$ -sentence has the same truth value in every existentially closed group (Corollary 9.6, page 121, [16]). So, all existentially closed groups simultaneously either \mathcal{K} -approximable or not. If there were a non \mathcal{K} -approximable group, then a finitely generated non \mathcal{K} -approximable group would exist. (\mathcal{K} -approximability is a local property.) One may embed this group into an existentially closed group E ([16], page 6 and proof of Theorem 3.10). As \mathcal{K} -approximability is closed under taking subgroups, E is not \mathcal{K} -approximable. So, all existentially closed groups are non \mathcal{K} -approximable. \square

Another application is related to algebraic groups over algebraically closed fields. The following statement is Proposition 20 of Section 5.

- *Any $\bar{w} \in Sys(Fin)$ is solvable in any algebraic group over an algebraically closed field.*

It implies

- *Let \mathcal{K} be a class of algebraic groups over algebraically closed fields (fields may be different for different groups from \mathcal{K}). Then \mathcal{K} -approximable groups are Fin -approximable, that is, weakly sofic.*

This corollary is a generalization of a result of [3]: “all linear sofic groups are weakly sofic”.

A subgroup $H < G$ has the congruence extension property (CEP) if any $N \triangleleft H$ satisfies $N = H \cap \langle\langle N \rangle\rangle_G$. A subgroup $H < G$ almost has CEP if there exists a finite set $\mathcal{F} \not\ni 1$ such

that any $N \triangleleft H$, $N \cap \mathcal{F} = \emptyset$, satisfies $N = H \cap \langle\langle N \rangle\rangle_G$. Some authors say that H is a normal convex subgroup of G if $H < G$ has the CEP, [7, 8, 27]. A group G is called SQ-universal if any countable group injects into a quotient of G .

Proposition (Some equivalence). *Let $F = \langle x, y \rangle$ be a free group of rank 2. Let $\mathcal{K} = \text{Nil}, \text{Sol}, \text{Fin}, \text{Alt}, \text{Sym}$ and $F \hookrightarrow \hat{F}_{\mathcal{K}}$ be pro-nilpotent, pro-solvable, pro-finite completion of F , \mathcal{A} or \mathcal{S} , respectively. Then the following are equivalent.*

1. All groups are \mathcal{K} -approximable.
2. F satisfies CEP in $\hat{F}_{\mathcal{K}}$.
3. F almost satisfy CEP in $\hat{F}_{\mathcal{K}}$.
4. $\hat{F}_{\mathcal{K}}$ is SQ-universal.

Similar proposition was proven by Goulnara Arzhantseva, Jakub Gismatullin and others [4]. It follows from construction of [17] that F does not possess CEP in its pro-nilpotent completion. In fact, results of [17] imply that any finitely generated perfect group is not *Nil*-approximable. The question “if all groups are *Sol*-approximable” seems to be open.

We use the following notations: $F < G$ denotes “ F is a subgroup of G ”. $F \triangleleft G$ denotes “ F is a normal subgroup of G ”. Let $X \subseteq F$ and $F < G$. Then $\langle X \rangle$ denotes “the subgroup generated by X ”; $\langle\langle X \rangle\rangle_F$ denotes “the normal subgroup of F , generated by X ”. If a group containing $\bar{a} = (a_1, \dots, a_k)$ is not assumed then $\langle \bar{a} \rangle$ denotes the free group freely generated by \bar{a} . As usual, $X \subset_{fin} Y$ means “ X is a finite subset of Y ”. For two sets A and B let $A \setminus B = \{a \in A \mid a \notin B\}$. Given $N \triangleleft G$, as usual, G/N denotes the quotient of G by N .

2 Approximation of groups

2.1 Metric approximation

In this subsection we define metric approximation, the $(\mathcal{K}, \mathcal{L})$ -approximation, of groups. We follow the lines of [2, 10, 24, 28].

Definition 2. *Let G be a group. An invariant (pseudo) length function is a map $\|\cdot\| : G \rightarrow [0, \infty[$ such that $\forall g, h \in G$*

- $\|1\| = 0$
- $\|gh\| \leq \|g\| + \|h\|$
- $\|h^{-1}gh\| = \|g\|$ ($\|\cdot\|$ is invariant within a conjugacy class)

(This is the definition of pseudo length function of [28]. The difference with the standard definition of the length function is not essential here, so we use just “length function”.)

Definition 3. *Let \mathcal{K} be a class of groups and \mathcal{L} be a class of invariant length functions on groups from \mathcal{K} . A group $G \in \mathcal{K}$ may have several length functions in \mathcal{L} . Let \mathcal{L}_G be the set of length functions on G in \mathcal{L} . All length functions are denoted by $\|\cdot\|$. We say that a group G is $(\mathcal{K}, \mathcal{L})$ -approximable if*

- there exists $\alpha : G \rightarrow \mathbb{R}$, $\alpha_1 = 0$ and $\alpha_g > 0$ for $g \neq 1$

- for any $\Phi \subset_{fin} G$, for any $\epsilon > 0$ there exist a function $\phi : \Phi \rightarrow H \in \mathcal{K}$ and $\|\cdot\| \in \mathcal{L}_H$ such that
 - $\phi(1) = 1$
 - $\|\phi(g)\| \geq \alpha_g$ for any $g \in \Phi$
 - $\|\phi(gh)(\phi(g)\phi(h))^{-1}\| < \epsilon$ for any $g, h, gh \in \Phi$.

Let $(\mathcal{K}, \mathcal{L}).approx$ denote the class of $(\mathcal{K}, \mathcal{L})$ -approximable groups.

Let ω be a non-principal ultrafilter over \mathbb{N} , $H_i \in \mathcal{K}$, and $\|\cdot\|_i \in \mathcal{L}_{H_i} \subset \mathcal{L}$. Let $N \triangleleft \prod_{i=1}^{\infty} H_i$ be defined as

$$(h_1, h_2, \dots) \in N \iff \lim_{\omega} \|h_i\|_i = 0.$$

Denote $\prod_{\omega} H_i = \prod_i H_i / N$, the metric ultraproduct of H_i (with respect to $\|\cdot\|_i$ and ω). The following characterization of $(\mathcal{K}, \mathcal{L}).approx$ is standard [10, 24, 28].

Proposition 1. *$G \in (\mathcal{K}, \mathcal{L}).approx$ if and only if there exists an injection $G \hookrightarrow \prod_{\omega} H_n$ for some non principal ultrafilter ω , a sequence $H_1, H_2, \dots \subset \mathcal{K}$, and a sequence of length functions $\|\cdot\|_i \in \mathcal{L}_{H_i}$.*

Definition 4. *Let $N \triangleleft G$. We say that N is $(\mathcal{K}, \mathcal{L})$ -separated normal subgroup of G if for some $\alpha : G \setminus N \rightarrow \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, for any $Y \subset_{fin} G \setminus N$, for any $\Phi \subset_{fin} N$, and for any $\epsilon > 0$ there exists a homomorphism $\phi : G \rightarrow H$, $H \in \mathcal{K}$, and $\|\cdot\| \in \mathcal{L}_H$ such that*

- $\|\phi(y)\| > \alpha_y$ for $y \in Y$,
- $\|\phi(x)\| < \epsilon$ for $x \in \Phi$.

The following proposition is a modification of Proposition 1.7 in [28], see also Lemma 6.2 of [13].

Proposition 2. *If N is a $(\mathcal{K}, \mathcal{L})$ -separated normal subgroup of G then G/N is $(\mathcal{K}, \mathcal{L})$ -approximable.*

Let F be a free group and $N \triangleleft F$. If F/N is $(\mathcal{K}, \mathcal{L})$ -approximable then N is a $(\mathcal{K}, \mathcal{L})$ -separated normal subgroup of F .

Remark. The difference of Proposition 2 with Proposition 1.7 of [28] is that in the last one the group G is assumed to be free. This assumption is not required for the first implication. In the second part of Proposition 2 the freeness of F is used to extend a map from generators of F to a homomorphism required by Definition 4.

2.2 \mathcal{K} -approximation

In this subsection we define the notion of \mathcal{K} -approximation which is independent of the length functions. In what follows we use the notation $x^g = g^{-1}xg$.

Definition 5. *Let G be a group and $X \subseteq G$. Let $C_n(X, G) = \{x_1^{g_1} x_2^{g_2} \dots x_n^{g_n} \mid x_i^{\pm} \in X, g_i \in G\}$ (the set of n -consequences of X in G). We just write $C_n(X)$ if the group G is uniquely assumed by a context.*

Let $X, Y \subset G$. We say that Y is n -separated from X (in G) if $Y \cap C_n(X, G) = \emptyset$.

Definition 6. Let \mathcal{K} be a class of groups. A group G is \mathcal{K} -approximable if for any $\Phi \subset_{fin} G$ for any $n \in \mathbb{N}$ there exist $H \in \mathcal{K}$ and a map $\phi : \Phi \rightarrow H$ such that

- $\phi(1) = 1$;
- the set $\phi(\Phi \setminus \{1\})$ is n -separated from $\{\phi(g)\phi(h)(\phi(gh))^{-1} \mid g, h, gh \in \Phi\}$.

Let $\mathcal{K}.approx$ denote the class of \mathcal{K} -approximable groups.

If G is \mathcal{K} -approximable (resp. $(\mathcal{K}, \mathcal{L})$ -approximable) then any subgroup of G is \mathcal{K} -approximable (resp. $(\mathcal{K}, \mathcal{L})$ -approximable). It is easy to verify and we will often use it without mention explicitly. There is the following relation between $(\mathcal{K}, \mathcal{L})$ -approximation and \mathcal{K} -approximation.

Lemma 3. If a group G is $(\mathcal{K}, \mathcal{L})$ -approximable then it is \mathcal{K} -approximable; if G is \mathcal{K} -approximable then one can define invariant length functions \mathcal{L} such that G is $(\mathcal{K}, \mathcal{L})$ -approximable.

Proof. Fix a group H with an invariant length function $\|\cdot\|$, $\epsilon > 0$ and $n \in \mathbb{N}$. Let $X = \{h \in H \mid \|h\| < \epsilon\}$ and $Y = \{h \in H \mid \|h\| \geq n\epsilon\}$. Then Y is n -separated from X . Comparing Definition 3 and Definition 6 gives us the first part of the Lemma.

Let $X, Y \subset H$ and Y be n -separated from X . Denote by $X^H = \{x^h \mid x^{\pm 1} \in X, h \in H\}$. The Cayley graph $\Gamma(H, X^H)$ defines a distance on H , $d(h_1, h_2) =$ 'the length of the shortest paths from h_1, h_2 in Γ or ∞ if there is no such a path'. It is easy to check that $\|h\| = \min\{\frac{1}{n}d(1, h), 1\}$ defines an invariant length function on H such that $\|x\| = 1/n$ for $x \in X$ and $\|y\| = 1$ for $y \in Y$. Now, one may convert a \mathcal{K} -approximation into a $(\mathcal{K}, \mathcal{L})$ -approximation where \mathcal{L} consists of the above constructed length functions. \square

We give analogues of Definition 4 and Proposition 2.

Definition 7. Let $N \triangleleft G$. We say that N is \mathcal{K} -separated normal subgroup of G if for any $n \in \mathbb{N}$, for any $Y \subset_{fin} G \setminus N$, and for any $\Phi \subset_{fin} N$ there exists a homomorphism $\phi : G \rightarrow H$, $H \in \mathcal{K}$, such that $\phi(Y)$ is n -separated from $\phi(\Phi)$.

Proposition 4. If N is a \mathcal{K} -separated normal subgroup of G , then G/N is \mathcal{K} -approximable.

Let $N \triangleleft F$ for a free group F . If F/N is \mathcal{K} -approximable then N is a \mathcal{K} -separated normal subgroup of F .

Let $G = \prod_{i \in \mathbb{N}} H_i$, be an (unrestricted) direct product. For $i \leq j$ let $\text{Pr}_i^j : G \rightarrow H_i \times H_{i+1} \times \dots \times H_j$ be the natural projection. Let $\text{Pr}_i = \text{Pr}_i^i$. We equip G with the product (or Tychonoff) topology. (In the most applications H_i are finite groups equipped with discrete topology.) For a class \mathcal{K} let $\prod(\mathcal{K})$ contains \mathcal{K} and all finite direct products of groups from \mathcal{K} .

Proposition 5. Suppose that \mathcal{K} is a class of compact groups. Let X be a closed subgroup of $G = \prod_{i \in \mathbb{N}} H_i$, $H_i \in \mathcal{K}$, such that $\text{Pr}_i(X) \in \mathcal{K}$ for any $i \in \mathbb{N}$. Let $N \triangleleft X$. Then X/N is $\prod(\mathcal{K})$ -approximable. (Note, that N need not to be topologically closed in G).

Proof. Let $\Phi \subset_{fin} X$ and $n \in \mathbb{N}$. Let $\Phi_N = \Phi \cap N$ and $\Phi_0 = \Phi \setminus N$. Notice that $C_n(\Phi_N, X) \cap \Phi_0 \subset N \cap \Phi_0 = \emptyset$. But

$$C_n(\Phi_N, X) = \bigcup_{\bar{f} \in \Phi_N^n} \{f_1^{x_1} f_2^{x_2} \dots f_n^{x_n} \mid x_i \in X\}$$

is a finite union of continuous images of a compact set X^n , hence $C_n(\Phi_N, X)$ is a compact set. So, there exists an open neighborhood U of Φ_0 such that $C_n(\Phi_N, X) \cap U = \emptyset$. It follows that

$$C_n(\Pr_1^k(\Phi_N), \Pr_1^k(X)) \cap \Pr_1^k(\Phi_0) = \Pr_1^k(C_n(\Phi_N, X)) \cap \Pr_1^k(\Phi_0) = \emptyset$$

for some $k > 0$ (the pre-images of open sets with respect to \Pr_1^k form the base of the Tychonoff topology). So, the homomorphisms $\Pr_1^k : X \rightarrow \Pr_1^k(X)$ satisfy Proposition 4 for the class $\prod(\mathcal{K})$. \square

Corollary 6. *Let \mathcal{K} be a class compact groups closed with respect of taking subgroups and finite direct products. Let X be a closed subgroup of a direct product of groups from \mathcal{K} . Then any quotient of X is approximable by \mathcal{K} . Particularly, it is true for $\mathcal{K} = \text{Nil}, \text{Sol}, \text{Fin}$, or class of all compact groups.*

Corollary 7. *Let $\mathcal{K} = \text{Alt}$ or $\mathcal{K} = \text{Sym}$. Then any quotient of a direct product $\prod_{i \in \mathbb{N}} H_i$, $H_i \in \mathcal{K}$ is \mathcal{K} -approximable.*

Proof. In Subsection 2.3 and Subsection 2.4 we show that the classes of Sym , $\prod(\text{Sym})$, Alt , $\prod(\text{Alt})$ -approximable groups coincide with the class of sofic groups. \square

2.3 Sofic groups

Classically, sofic groups are defined using metric approximations. In this subsection we show that we can avoid the use of the length functions. Let S_m denote the group of all permutations of $[m] = \{1, 2, \dots, m\}$ (symmetric group on m elements); $A_m \triangleleft S_m$ denote alternating group on m elements. The normalized Hamming length function $\|\cdot\|$ is, by definition,

$$\|h\| = \frac{|\{x \in [m] \mid xh \neq x\}|}{m},$$

here we suppose that $h \in S_m$ has a right natural action, $h : x \rightarrow xh$, on $[m]$. It is defined on S_m as well as on A_m for any $m \in \mathbb{Z}^+$. Let \mathcal{H} denote the class of all normalized Hamming length functions.

Definition 8. *$(\text{Sym}, \mathcal{H})$ -approximable groups are said to be sofic groups.*

Lemma 8. *$(\text{Alt}, \mathcal{H})$.approx coincides with the class of all sofic groups.*

Proof. $A_m < S_m$. So any $(\text{Alt}, \mathcal{H})$ -approximation is a $(\text{Sym}, \mathcal{H})$ -approximation. It proves that $(\text{Alt}, \mathcal{H})$.approx \subseteq $(\text{Sym}, \mathcal{H})$.approx. On the other hand, $S_m \hookrightarrow A_{2m}$ (just repeating each cycle of a permutation twice). This inclusion preserve $\|\cdot\|$. So, any $(\text{Sym}, \mathcal{H})$ -approximation may be converted into an $(\text{Alt}, \mathcal{H})$ -approximation. It shows that

$$(\text{Sym}, \mathcal{H})\text{.approx} \subseteq (\text{Alt}, \mathcal{H})\text{.approx}.$$

\square

Proposition 9. *$\text{Sym}.\text{approx} = \prod(\text{Sym}).\text{approx} = \text{Alt}.\text{approx} = \prod(\text{Alt}).\text{approx} =$ "sofic groups".*

We will proof the proposition in subsection 2.4. Now we demonstrate some auxiliary results. The following lemma is an analogue of Lemma 2.5 of [12] with a similar proof.

Lemma 10. *Let A be an alternating group, $X \subset A$ and $y \notin C_n(X, A)$. Then $\frac{\|y\|}{\|x\|} \geq \frac{n-1}{16}$ for any $1 \neq x \in X$. In other words, denoting $\epsilon = \sup\{\|x\| \mid x \in X\}$ one gets $\{\alpha \in A \mid \|\alpha\| < \frac{n-1}{16}\epsilon\} \subset C_n(X, A)$.*

Proof. This lemma is a manifestation of the fact that in a finite simple group powers of a conjugacy class cover the group “almost as fast as possible” [20]. The case of alternating groups was considered in [6]. Let C_x denote the conjugacy class of $x \neq 1$ in A . Lemmas 2.05, 2.06, 3.03 of [6] imply that C_x^4 contains all nontrivial even permutations of support of x . (Support of x are elements which are not fixed by x .) Then we may shift support by conjugation and construct $r = \lceil \frac{\|y\|}{\|x\|} \rceil$ permutations with supports partitioning the support of y . So, C_x^{4r} contains a permutation with the same support as y . Applying ones again Lemmas 2.05, 2.06, 3.03 of [6] we obtain that $y \in C_x^{16r}$. \square

Remark. It looks like that one may change $\frac{n-1}{16}$ by $\frac{n-1}{4}$ in the lemma. For this we should start with r shifts of x and cover the support of y , but we need some extra details if the support of y is almost all set $[m] \dots$

Lemma 11. *Let G be a group, $X \subset G$, and $y \in G$. Suppose that there exists a homomorphism $\phi : G \rightarrow A_m$, such that $\|\phi(y)\|/\|\phi(x)\| > n \geq 2$ for any $x \in X$. Then there exists $r = r(y)$ and a homomorphism $\psi : G \rightarrow A_{mr}$, such that $\|\psi(y)\| \geq 1/2$ and $\|\psi(x)\| < 1 - (1/4)^{1/n}$.*

Proof. If $\|\phi(y)\| \geq 1/2$, put $r = 1$ and $\psi = \phi$. We are done. If $\|\phi(y)\| < 1/2$ the arguments are based on the amplification trick, see [12]. To make exposition self-contained we explain it here. Define an inclusion $h \rightarrow h^{\otimes r} : A_m \rightarrow A_{mr}$ as follows. Consider A_{mr} as even permutations of $[m]^r$. For $(j_1, j_2, \dots, j_r) \in [m]^r$ let $(j_1, j_2, \dots, j_r)h^{\otimes r} = (j_1h, j_2h, \dots, j_rh)$. It is easy to check that $1 - \|h^{\otimes r}\| = (1 - \|h\|)^r$.

Now, let $\psi(z) = \phi^{\otimes r}(z)$, then $1 - \|\psi(z)\| = (1 - \|\phi(z)\|)^r$. We may choose $r \in \mathbb{N}$ such that $1/4 < 1 - \|\psi(y)\| \leq 1/2$. We just need to estimate $\psi(x)$ for $x \in X$:

$$\frac{\log(1 - \|\psi(y)\|)}{\log(1 - \|\psi(x)\|)} = \frac{\log(1 - \|\phi(y)\|)}{\log(1 - \|\phi(x)\|)} \geq \frac{\|\phi(y)\|}{\|\phi(x)\|} > n.$$

So, the estimate for $\|\psi(x)\|$ follows. \square

The following lemma is a strengthening of Proposition 2 for sofic groups.

Lemma 12. *Let G be a group and $N \triangleleft G$. Suppose, that for any $y \in G \setminus N$, any $\Phi \subset_{fin} N$, and any $\epsilon > 0$, there exists a homomorphism $\phi : G \rightarrow A_m$ such that $\|\phi(y)\| \geq 1/2$ and $\|\phi(x)\| < \epsilon$ for any $x \in \Phi$. Then G/N is sofic.*

Proof.

Claim 13. *Let G be a group, $X \subset G$, $Y \subset_{fin} G$, and $\epsilon > 0$. Suppose, that for any $y \in Y$ there exists a homomorphism $\phi_y : G \rightarrow A_{m_y}$ such that $\|\phi_y(y)\| \geq 1/2$ and $\|\phi_y(x)\| < \epsilon$ for any $x \in X$. Then there exists $\psi : G \rightarrow A_m$ such that $\|\psi(y)\| \geq \frac{1}{2|Y|}$ for any $y \in Y$ and $\|\psi(x)\| < \epsilon$ for any $x \in X$.*

Proof. For $\alpha \in A_r$ and $\beta \in A_k$ we define $\alpha \oplus \beta \in A_{r+k}$. Consider A_{r+k} as permutation groups on disjoint union $[r] \dot{\cup} [k]$. Then

$$x(\alpha \oplus \beta) = \begin{cases} x\alpha & \text{if } x \in [r] \\ x\beta & \text{if } x \in [k] \end{cases}$$

It is clear that $\|\alpha \oplus \beta\| = \frac{r\|\alpha\| + k\|\beta\|}{r+k}$. Let $r = \text{gcm}(\{m_y \mid y \in Y\})$. Replacing each ϕ_y by the sum $\phi_y \oplus \phi_y \dots (r/m_y \text{ copies of } \phi_y)$ we may assume that $\phi_y : G \rightarrow A_r$ for every $y \in Y$. Now, take $\psi(z) = \bigoplus_{y \in Y} \phi_y(z)$. \square

In order to prove the lemma we use Proposition 2. Fix $Y \subset_{fin} G \setminus N$, $\Phi \subset_{fin} N$, $\epsilon > 0$ we show that the hypothesis of the Lemma guarantee the assumptions of Proposition 2 for $\alpha_y = 1/3$. Choose sufficiently small $0 < \epsilon'$. For every $y \in Y$ consider $\phi_y : G \rightarrow A$ such that $\|\phi_y(y)\| \geq 1/2$ and $\|\phi_y(x)\| < \epsilon'$ for $x \in \Phi$. Then construct ψ of Claim 13. Applying the amplification trick to ψ we obtain ϕ of Proposition 2. Notice, that the choice of ϵ' depends on ϵ and $|Y|$ only. \square

2.4 Proof of Proposition 9

Let us start with *Alt* parts of the proposition. Notice, that 'sofic groups' $= (\text{Alt}, \mathcal{H}).\text{approx} \subseteq \text{Alt}.\text{approx} \subseteq \prod(\text{Alt}).\text{approx}$ by Lemma 3 and Lemma 8. So, it suffices to show that $\prod(\text{Alt})$ -approximable groups are $(\text{Alt}, \mathcal{H})$ -approximable. Let $G = F/N$ be $\prod(\text{Alt})$ -approximable, F be a free group. Let $y \in F \setminus N$ and $\Phi \subset_{fin} N$. Fix $\epsilon > 0$ and $n \in \mathbb{N}$ such that $\epsilon < 1 - (1/4)^{\frac{16}{n-1}}$. By Proposition 4 chose a homomorphism $\phi : F \rightarrow \tilde{A} = \prod_{j=1}^k A_j$ such that $\{\phi(y)\}$ is n -separated from $\phi(\Phi)$. Let $\epsilon_j = \max\{\|\text{Pr}_j(\phi(x))\| \mid x \in \Phi\}$ and $D_j = \{y \in A_j \mid \|y\| < \frac{n-1}{16}\epsilon_j\}$. As $\text{Pr}_j(C_n(X, \tilde{A})) = C_n(\text{Pr}_j(X), A_j)$, by Lemma 10 we get $\prod_{j=1}^k D_j \subset C_n(\phi(\Phi), \tilde{A})$. So, $\text{Pr}_j\phi(y) \notin D_j$ for some j . The *Alt*-part of the proposition follows by Lemmas 11 and 12.

Let G be a $\prod(\text{Sym})$ -approximable group. By Proposition 1 and Lemma 3 G is a subgroup of P , a quotient of a direct product of symmetric groups. Symmetric groups are extensions of alternating groups by the cyclic group with 2 elements. So, P is an extension of Q , a quotient of a direct product of alternating groups, by an abelian group. As the proposition and Corollary 7 is proven for *Alt*, the group Q is sofic. Now, P is sofic as an extension of Q by amenable group, see [12].

3 Γ -groups.

Definition 9. Let Γ be a group. A group G with injective homomorphism $\Gamma \hookrightarrow G$ is said to be an Γ -group. One may think that G has a fixed copy of $\Gamma < G$. Let G_1, G_2 be Γ -groups. A homomorphism $\phi : G_1 \rightarrow G_2$ is called a Γ -morphism if the restriction of ϕ on Γ is the identity map, $\phi|_\Gamma = \text{Id}_\Gamma$. Notation $\phi : G_1 \rightarrow_\Gamma G_2$ means ϕ is a Γ -morphism.

This definition has an important use in algebraic geometry over groups, [5]. Clearly, Γ -groups form a category. For an Γ -group G and $N \triangleleft \Gamma$ we denote $\bar{N}^G = \Gamma \cap \langle\langle N \rangle\rangle_G$. It is clear that \cdot^G is an idempotent. The condition $N = \bar{N}^G$ is equivalent to the existence of the

following commutative diagram with injective ψ :

$$\begin{array}{ccc} \Gamma & \xrightarrow{\phi} & G \\ \downarrow & & \downarrow \\ \Gamma/N & \xrightarrow{\psi} & H \end{array} \quad (1)$$

In the category of Γ -groups not for all pairs of objects G_1, G_2 there exists a morphism $G_1 \rightarrow_{\Gamma} G_2$. (For example, G_1 is a simple group, Γ is a proper subgroup of G_1 and $G_2 = \Gamma$.)

Lemma 14. *Suppose that there exists Γ -morphism $H \rightarrow_{\Gamma} G$. Then $\bar{N}^H \subseteq \bar{N}^G$ for any $N \triangleleft \Gamma$.*

Proof. Let $\phi : H \rightarrow_{\Gamma} G$ be a Γ -morphism. Then

$$\bar{N}^G = \langle\langle N \rangle\rangle_G \cap \Gamma \supseteq \langle\langle \phi(N) \rangle\rangle_{\phi(H)} \cap \phi(\Gamma) = \phi(\langle\langle N \rangle\rangle_H) \cap \phi(\Gamma) \supseteq \phi(\langle\langle N \rangle\rangle_H \cap \Gamma) = \bar{N}^H$$

□

Let G be a Γ -group. Denote

$$\text{FP}(G) = \{H : H \text{ is a finitely presented } \Gamma\text{-group and } \exists \phi : H \rightarrow_{\Gamma} G\},$$

here we call a Γ -group H finitely presented if it is finitely presented relative to Γ , that is, $H = (\Gamma * F)/M$, where F is a free group of finite rank and $M \triangleleft (\Gamma * F)$ is finitely generated (as a normal subgroup).

Lemma 15. *Let G be a Γ -group and $N \triangleleft \Gamma$. Then*

$$\bar{N}^G = \bigcup_{H \in \text{FP}(G)} \bar{N}^H$$

Proof. Lemma 14 shows that \bar{N}^G contains $\bigcup_{H \in \text{FP}(G)} \bar{N}^H$. Now, suppose, that $w \in \bar{N}^G$.

Consider $G = \Gamma * F_{\infty}/M$, where F_{∞} is a free group of countable rank. Then w may be presented in $\Gamma * F_{\infty}$ as a finite product $w = w_1 w_2 \dots w_r$, where each w_i either in M or equal to a conjugate (in $\Gamma * F_{\infty}$) of some element of N . Let $F_k < F_{\infty}$ be a free group generated by all elements of F_{∞} appearing in w_i , $i = 1, \dots, r$. Let $\tilde{M} \triangleleft \Gamma * F_k$ be generated by $\{w_1, \dots, w_r\} \cap M$. Notice, that $H = (\Gamma * F_k)/\tilde{M}$ is a Γ -group (as it has less relations then G). It follows that $H \in \text{FP}(G)$ and $w \in \bar{N}^H$. □

Recall, that $\langle \bar{a} \rangle$ denotes the free group generated by $\bar{a} = a_1, \dots, a_r$. Let \mathcal{K} be a class of finite groups. Consider the set of all homomorphisms $\phi : \langle \bar{a} \rangle \rightarrow H$, for all $H \in \mathcal{K}$. This set is countable and we may enumerate it, say $\phi_i : \langle \bar{a} \rangle \rightarrow H_i$. Let $\mathcal{K}(\langle \bar{a} \rangle) = \prod_{i=1}^{\infty} H_i$. Then ϕ_i define homomorphism $\phi_{\infty} : \langle \bar{a} \rangle \rightarrow \mathcal{K}(\langle \bar{a} \rangle)$, $\phi_{\infty}(w) = (\phi_1(w), \phi_2(w), \dots, \phi_j(w), \dots)$. A free group $\langle \bar{a} \rangle$ is residually \mathcal{K} if ϕ_{∞} is an injection. Let us denote by $\mathcal{K}'(\langle \bar{a} \rangle)$ the closure of $\phi_{\infty}(\langle \bar{a} \rangle)$ with respect to the product topology.

Theorem 16. *Let $\mathcal{K} \subseteq \text{Fin}$ such that $\mathcal{K}.\text{approx} = \prod(\mathcal{K}).\text{approx}$. Let a free group $\langle \bar{a} \rangle$ be residually \mathcal{K} . Consider $\mathcal{K}(\langle \bar{a} \rangle)$ as an $\langle \bar{a} \rangle$ -group. Then $\langle \bar{a} \rangle/N$ is \mathcal{K} -approximable if and only if $N = \bar{N}^{\mathcal{K}(\langle \bar{a} \rangle)}$.*

Suppose, in addition, that \mathcal{K} is closed under taking subgroups. Then $\langle \bar{a} \rangle/N$ is \mathcal{K} -approximable if and only if $N = \bar{N}^{\mathcal{K}'(\langle \bar{a} \rangle)}$.

Proof. We start with the first part of the theorem.

\Rightarrow . Let $\langle \bar{a} \rangle / N$ be \mathcal{K} -approximable. Suppose, searching for a contradiction, that there exists $y \in \langle \bar{a} \rangle \setminus N$ such that $y \in \langle \langle N \rangle \rangle_{\mathcal{K}(\langle \bar{a} \rangle)}$. It follows that $y \in C_n(X, \mathcal{K}(\langle \bar{a} \rangle))$ for some $n \in \mathbb{N}$ and some $X \subset_{fin} N$. Using definition of $\mathcal{K}(\langle \bar{a} \rangle)$ and applying projection on H_i we get $\phi_i(y) \in C_n(\phi_i(X), H_i)$. A contradiction with Proposition 4 as (H_i, ϕ_i) run over all pairs $H \in \mathcal{K}$, $\phi : \langle \bar{a} \rangle \rightarrow H$.

\Leftarrow . Let $N = \bar{N}^G$. By Diagram (1) we see that $\langle \bar{a} \rangle / N$ is a subgroup of $Q = G / \langle \langle N \rangle \rangle_G$. Q is $\prod(\mathcal{K})$ -approximable by Proposition 5. We are done by the hypothesis $\mathcal{K}.approx = \prod(\mathcal{K}).approx$ and the fact that subgroups of \mathcal{K} -approximable groups are \mathcal{K} -approximable.

The second part of the theorem. Let $G = \mathcal{K}(\langle \bar{a} \rangle)$ and $H = \mathcal{K}'(\langle \bar{a} \rangle)$. As $H \hookrightarrow_{\langle \bar{a} \rangle} G$ one has $\bar{N}^H \subseteq \bar{N}^G$ by Lemma 14. So, if $\langle \bar{a} \rangle / N$ is \mathcal{K} -approximable then $N \subseteq \bar{N}^H \subseteq \bar{N}^G = N$ by the first part of the theorem. On the other hand, if $\bar{N}^H = N$ then $\langle \bar{a} \rangle / N \hookrightarrow H / \langle \langle N \rangle \rangle_H$. By Proposition 5 $H / \langle \langle N \rangle \rangle_H$ is $\prod(\mathcal{K})$ -approximable as well as all of its subgroups. We are done by the hypothesis $\mathcal{K}.approx = \prod(\mathcal{K}).approx$.

Remark In fact, it is possible to show that $\bar{N}^H = \bar{N}^G$. Indeed, under assumption of the theorem there exists a $\langle \bar{a} \rangle$ -morphism $G \rightarrow_{\langle \bar{a} \rangle} H$. \square

4 Soficity and equations over groups

Let us consider the concepts given in Definition 1 of the introduction. A system from $Sys(\mathcal{K})$ is said to be a \mathcal{K} -system. It is clear, that if a system is solvable in G then it is solvable in any quotient of G . Also, any \mathcal{K} -system is solvable in any direct product of groups from \mathcal{K} . Using Proposition 1 and Lemma 3 we obtain the following proposition, see [24].

Proposition 17. *If G is \mathcal{K} -approximable then any \mathcal{K} -system is solvable over G .*

The following lemma is trivial.

Lemma 18. $\langle a_1, \dots, a_r, x_1, \dots, x_n \mid \bar{w}(\bar{a}, \bar{x}) \rangle \in FP(\mathcal{K}(\langle \bar{a} \rangle))$ if and only if $\bar{w} \in Sys(\mathcal{K})$

This lemma, Theorem 16, and Proposition 17 yield the following corollary.

Corollary 19. *Let $\mathcal{K} \subseteq Fin$ and $\mathcal{K}.approx = \prod(\mathcal{K}).approx$. Then a group G is \mathcal{K} -approximable if and only if any \mathcal{K} -system is solvable over G .*

5 Algebraic groups

Here we study algebraic groups over algebraically closed fields (AGCF).

Proposition 20. *Any $\bar{w} \in Sys(Fin)$ is solvable in an AGCF group.*

Proof. The proof is very similar to the proof of Malcev theorem in [9], see also [3].

Let G be an algebraic group over algebraically closed field K . Let $\mathcal{D} \subset K[\bar{z}]$ be the defining equations $\bar{g} \in G^r$. Suppose, that $\bar{w}(\bar{g}, \bar{x})$ is a system without solution in G . Let $\mathcal{S} \subset K[\bar{z}]$ be the algebraic equations defined by \bar{w} . By Hilbert's Nullstellensatz $1 \in I(\mathcal{S})$, where $I(X)$ is an ideal generated by X . This means that there exist $f_1, f_2, \dots, f_k \in K[\bar{z}]$, and $h_1, h_2, \dots, h_k \in \mathcal{S}$ such that

$$1 = \sum_{i=1}^k g_i h_i. \quad (2)$$

Let B be the set of elements of K appearing in $g_i, h_i, \mathcal{D}, \mathcal{S}$. The set B is finite. Consider the ring $R \subset K$, generated by B . Clearly, Eq.(2) holds in $R[\bar{z}]$ and no nontrivial homomorphism $R \rightarrow R/I$ sends 1 to 0. Now, there exists a maximal ideal I of R . The corresponding field R/I is finite [9]. Let $\phi : R \rightarrow R/I$ be the natural morphism. This morphism define a group homomorphism $\phi : G \rightarrow \phi(G)$. Then the system $\phi(\mathcal{S})$ does not have solutions in R/I and $\bar{w}(\phi(\bar{g}), x)$ does not have solutions in the finite algebraic group $\phi(G)$. \square

Corollary 21. *Let A be a quotient of a direct product of AGCF groups. Suppose that $G < A$. Then the group G is Fin-approximable, that is, G is weakly sofic.*

Corollary 22. *Let \mathcal{K} be a class of AGCF groups. Then $\mathcal{K}.\text{approx} \subset \text{Fin}.\text{approx}$.*

Notice, that the proof of Proposition 20 essentially uses the algebraic closeness. For example, $x^2 + y^2 = -1$ has no solutions in \mathbb{R} , but has a solution in any finite field. The hyperlinear groups are defined as being approximable by finite dimensional unitary groups (with normalized Hilbert-Schmidt norm as a length function) [10, 24]. So, the following is an open question.

Open Question 23. *Are hyperlinear groups weakly sofic?*

6 Proof of Proposition (Some equivalence).

(1) \implies (2). This is an implication of Theorem 16.

(2) \implies (3). This is trivial.

(3) \implies (4). Given a finite set \mathcal{F} , using small cancellation theory we may find $H < F$ such that $\langle\langle H \rangle\rangle_F$ avoids $\mathcal{F} \not\ni 1$ and H has CEP in F . It follows that \hat{F} has a free subgroup with CEP. So, \hat{F} is SQ-universal. Explicitly, let $A_i = a^{100}b^i a^{101}b^i \dots a^{199}b^i$. Then $H = \langle A_i, A_{i+1} \rangle$ satisfies our properties for i sufficiently large by results of [23].

(4) \implies (1). As $\bar{w} \in \text{Sys}(\mathcal{K})$ has a solution in any quotient of \hat{F} , the system \bar{w} is solvable over any (countable) group G . And we are done by Corollary 19.

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